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On the  
Cayley-Veronese Class of  
Configurations

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On the Cayley-Klein Class of  
Configurations.

Cayley, in his paper, Sur  
quelques theoremes de la geo-  
metrie de position (Cullis Jour-  
nal, Vol. 31, 1846; also Collected  
Works, Vol. 1, p. 317), first calls  
attention to the figures ob-  
tained by taking  $n$  points  
in a flat space of any num-  
ber of dimensions, say  $v$ ,  
joining each two points  
by a line, each three by a  
plane, . . . . . each  $v$  by a  
flat space of  $v-1$  dimensions,  
and then taking the section  
by a plane or three-dimen-



sional space.

Thirty years later, however, in this manner, Abel has the projective relationships of space of various dimensions by the principle of projection and section (Math. Ann., Vol. 19, 1882) discussed more fully the nature of this class of configurations, treating not only the configurations then obtained in the plane and in three-dimensional space, but, more generally, those in space of  $r$  dimensions. Both Cayley and Salmon state that these same configurations can also be ob-



tained as projections of higher dimensional figures.

In a paper, Sulla poliedrale Configurazioni (Math. Ann., Vol. 34, 1887), de Vries discusses the configurations which are plane sections of the complete  $n$ -point in ordinary three-dimensional space.

Among these papers of Caporali which were not published till after his death, is a paper Sulla teoria della configurazioni (Memorie di Geometria, p. 262) written in 1877, giving, without proof, a number of theorems concerning a str-



tain class of configuration  
in the plane. Although there  
is no mention of higher-  
dimensional space in Coper-  
nic's paper, his configura-  
tions are the same as those  
obtained by Cayley and De-  
rourne as sketches of higher-  
dimensional figures.

It is only purpose in  
this paper to prove Copernic's  
theorem by regarding  
his configurations as sections  
or projections of higher-dimen-  
sional figures, to give simi-  
lar theorems for the three-  
dimensional configurations,  
and finally to give a con-  
struction of the quadrics



poler system — the plane  
and the space bound upon  
certain of these configurations.

Instances of the formation  
of configurations of this class  
in connection with other sub-  
jects are numerous.

The well known Desargues  
configuration<sup>1</sup> of two perspective  
triangles with their centers and  
axis of perspective, treated in  
the aforementioned paper of

<sup>1</sup> Discussed by Desargues in  
a paper, Exemple de l'usage des  
moyens raisonnables de G. G. G. G.  
Touchant la pratique de la perspec-  
tive. (1636) Cf. Écrits, Bibliothèque  
MATH., 1845.



Cayley and also in Klein's  
paper in the (S. 22), Sitzungsberichte der Math. Akad.,  
Vol. 14, 1881, belongs to this  
class. Teronese shows in  
his memoirs, Memorie teronese  
sull' hexagramma magico  
(Atti della R. Acad. di Lincei,  
Vol. 1, 1877) and Interpretation  
géométrique de la théorie des  
substitutions de six lettres en  
relation avec les propriétés de  
l'hexagramme magique (Annali  
di Mat., Vol. 11, 1881), that  
the sixty lines of the Pascal  
hexagon form six of these  
hexagrammic figures.

The configuration of  
two perspective tetrahedra with



their center and plane of perspective occurs in Klein's memoir, *Über die Erzeugnisse des projektiven Raumes* (Math. Ann., Vol. 2, 1870); and also in Pich's paper, *Über die Figuren von sechs Punkten im Räume von vier Dimensionen* (Ann. Jour. of Math., Vol. 31, 1900).

Whithead, from a different standpoint, treats the figure of two perspective "referred figures" in space of any number of dimensions (Abstract Algebra, Vol. 1, p. 107).

Finally, in the paper to which I have already referred, above that by Pich,



when applied one of the  
plane figure of fifteen lines  
and twenty points obtained  
as a section of the figure of  
two perspective tetrahedra, de-  
rived in the Pappus hexagon.

There are only a few of  
the many instances which  
might be cited of the re-  
verse of configuration of  
this class.

In this paper I shall  
use the conventional  $S_1$  to  
represent a line or flat  
space of a dimension 1 (or  
point). I shall occa-  
sionally use the words "a point,"  
"a plane", etc., to signify the  
dual of the point or plane, etc.



in the space in which I am  
 operating. Thus, in  $S_n$ , a  
 co-point is in  $S_{n-1}$ , a co-plane  
 is in  $S_{n-2}$ , etc. By a  
complete co-point I shall  
 mean the figure of  $n$  points  
 together with the lines deter-  
 mined by the points two at  
 a time, the planes determined  
 by the points three at a time,  
 ----- the  $S_{n-1}$ 's determined by  
 the points  $n$  at a time. The  
 complete co-point is the  
 dual figure. I shall  
 use the symbol  $\binom{n}{r}$  to re-  
 vote the number of combina-  
 tions of  $n$  things taken  $r$  at  
 a time, adopting the conven-  
 tion  $\binom{0}{0} = \binom{n}{n} = 1$



Let  $T_{n,r}^v$  denote the con-  
figuration obtained in  $S_r$   
by cutting a complete  $n$ -  
point in  $S_v$  by the  $S_r$ . Let  
each of the  $n$  points in  $S_v$   
be denoted by a letter. The  
 $T_{n,r}^v$  consists, as Peroneau shows,<sup>1</sup> of  
 $\binom{n}{v-r+1}$  points, each named by  $v-r+1$  of the letters,  
 $\binom{n}{v-r+2}$  lines, " " " " " " " " " " " "  
" " " " " " " " " " " "  
" " " " " " " " " " " "  
 $\binom{n}{v-1}$   $S_{r-2}$ 's, " " " " " " " " " " " "  
and  $\binom{n}{v}$   $S_{r-1}$ 's, " " " " " " " " " " " "

Any  $S_p$  of  $T_{n,r}^v$  is then named  
by  $p+v-r+1$  letters. Any two of

<sup>1</sup> Math. Ann., Vol. 19, p. 111;  
also Sammlung der Geometrie, p. 215



elements, say  $S_p$  and  $S_q$  ( $p > q$ ), if the  $\Gamma_{q,r}$  are incident if the  $q+r-1$  letters of the  $S_q$  are contained among the  $p+r-1$  letters of the  $S_p$ . It follows that there are

$$\left. \begin{array}{l} v \quad S_{r-2}'s \\ (v-2) \quad S_{r-3}'s \\ \vdots \\ (p+v-r+1) \quad S_p's \end{array} \right\} \text{ on each } S_{r-1};$$

$$\left. \begin{array}{l} (n-v+r-1) \quad S_{r-1}'s \\ (n-v+r-1) \quad S_{r-2}'s \\ \vdots \\ (n-v+r-1) \quad S_p's \end{array} \right\} \text{ on each point};$$

etc.

Suppose now we start with the complete  $n$ -subset  $S_v$  which consists of



$n$   $S_{r-1}$ 's, each named by a letter,  
 (2)  $S_{r-2}$ 's, each named by 2 of the  $n$  letters,

$\vdots$   
 (r-1) lines,  $\dots \dots \dots v_{r-1} \dots \dots \dots$

(r) points,  $\dots \dots \dots v \dots \dots \dots$

Any  $S_p$  of this figure is named by  $v$ - $p$  letters; and an  $S_p$  and  $S_q$  ( $p > q$ ) of the figure are incident if the  $v$ - $p$  letters of the  $S_p$  are contained among the  $v$ - $q$  letters of the  $S_q$ .

Let us now project this figure upon an  $S_r$ , taking as  $o$ - $S_r$ , or  $S_{r-r+1}$ , as the projection-centre. By such a projection,<sup>1</sup> each point, line,

<sup>1</sup> This is what Veronese calls an incident projec-



plane,  $S_1, \dots, S_{r-1}$  of the figure is sent into a point, line, plane,  $S_1, \dots, S_{r-1}$ , in the  $S_r$ ; for any  $S_p$  of the figure is joined to the  $S_{r+1}$  by an  $S_{r+p}$ , and this  $S_{r+p}$  is cut by the  $S_r$  in an  $S_p$ . We name each element of the new figure in  $S_r$  by the name of that element of the original figure from which it was derived. The  $S_r$ 's,  $S_{r+1}$ 's,  $\dots$ ,  $S_{r-1}$ 's of the original figure are lost in the pro-

tion. Cf. Grundzüge der Geometrie  
p. 612



jection.

Notice also that an  $S_p$  and  $S_q$  which are incident in the original figure are projected into an  $S_p$  and  $S_q$  which are incident. For the original  $S_p + S_q$  will be joined to the  $S_{v-r+1}$  by an  $S_{v-r+p}$  and an  $S_{v-r+q}$  which are incident, and these will be cut by the  $S_r$  in an  $S_p$  and an  $S_q$  which are incident.

The figure obtained in the  $S_r$  will then consist of  
( $\overset{r}{v-r+1}$ )  $S_{r-1}$ 's, each named by  $v-r+1$  letters,  
( $\overset{r}{v-r+2}$ )  $S_{r-2}$ 's, " " " "  
⋮  
( $\overset{r}{v-1}$ ) lines, " " " "  
and ( $\overset{r}{v}$ ) points, " " " "



and any  $S_p$  and  $S_q$  if this fig  
are not incident if the index  
of the  $S_p$  is contained in the  
index of the  $S_q$ . It follows  
that there are

$$\left. \begin{array}{l} (r-2) \text{ planes} \\ \vdots \\ (r-p) S_p's \end{array} \right\} \text{ on each point};$$

$$\left. \begin{array}{l} (n-v+r-1) \\ (r-1) \text{ points} \\ \vdots \\ (n-v+r-1) S_p's \end{array} \right\} \text{ on each } S_{r-1};$$

etc.

This configuration, which  
we will denote by  $\Sigma_{n,r}^{v,1}$  is evi-  
dently the dual of the  $\Gamma_{n,r}^{v,0}$ .

I wish to call atten-  
tion to the fact that the  
projection of a figure in  $S_r$



upon an  $S_r$  need not be per-  
 formed at a single stroke, but  
 may be accomplished by a  
 number of steps. We may  
 first project upon an  $S_{r_1}, \alpha_1$ ,  
 from a point,  $a_1$ . I shall  
 speak of this as a simple  
projection. Our figure now  
 lies in an  $S_{r_1}$ ; and, confining  
 ourselves to this  $S_{r_1}$ , we project  
 just again upon an  $S_{r_2}, \alpha_2$ ,  
 from a point,  $a_2$ . Contin-  
 uing the process, we would,  
 after  $r$  such simple projec-  
 tions, obtain a figure in an  
 $S_r$ . This figure will be pre-  
 cisely the same as that ob-  
 tained by projecting at once  
 upon the  $S_r$  from an  $S_{r+r-1}$ .



provided  $x_1, x_2, \dots, x_{r-1}$  are all  
 chosen incident with the  $S_{r-1}$   
 and  $x_1, x_2, \dots, x_{r-1}$  all inci-  
 dent with the  $S_r$ . Moreover,  
 we may make any number  
 of simple projections, say on  
 $\{m < m-r\}$ , obtaining a figure  
 in  $S_{m-r}$ , and then project  
 from an  $S_{m-r+1}$  upon an  $S_r$ .

Any element,  $S_p$ , of a  
 $C_{n,r}^v$  is named by a combina-  
 tion of  $v-p$  letters out of  $n$ .  
 If we rename all the elements,  
 designating as a new name  
 to each  $S_p$  the  $n-v+p$  letters  
not contained in the old  
 name, we see at once that  
 the  $C_{n,r}^v$ , under this new let-  
 tering, becomes a  $T_{n,r}^{n-v+r-1}$ . Q



$\Gamma_{n,r}^v$ , with a similar change of lettering becomes a  $C_{n,r}^{m-v+r-1}$ . (I shall, for convenience, write  $m$  for  $m-v+r-1$ .) Since a  $\Gamma_{n,r}^v$  and a  $C_{n,r}^v$  are dual figures, it follows that a  $C_{n,r}^v$  and a  $C_{n,r}^m$  are dual, as also a  $\Gamma_{n,r}^v$  and a  $\Gamma_{n,r}^m$ . Also a  $C_{n,r}^v$  or a  $\Gamma_{n,r}^v$  is self-dual if  $m=v$ , i.e., if  $2v=n+r-1$ .

While the symbols  $C_{n,r}^v$  and  $\Gamma_{n,r}^m$  represent the same configuration, I shall have occasion to use them both. I shall use  $\Gamma_{n,r}^m$  when I wish to regard the configuration as the section of a  $m$ -dimensional figure, and  $C_{n,r}^v$  when I wish to regard it as the projection



of a  $v$ -dimensional figure.

Caporali<sup>2</sup> defines an "elementary configuration (in the plane)† of order  $n$  and class  $v$ " (which he denotes by the symbol  $C_n^v$ ) as a systematic arrangement of  $\binom{n}{v}$  points and  $\binom{n}{v-1}$  lines, in which it is possible to name each point by a combination of  $v$  letters out of  $n$ , and each line by a combination of  $v-1$  letters out of  $n$ , in such a way that the points on a line are obtained by adding successively to the name of the line each letter not in

<sup>2</sup>

Memoria di Geometria, § 222





bdeac  
 bdaec cdeab  
 cdone  
 bcead  
 ceahd  
 acebd  
 beacd  
 aebrd  
 abecd  
 adebc  
 adbec  
 abcrd  
 abdce  
 acdbe  
 bcade  
 abcde

Figure 1.  
 (To face page 20.)

tained in it, and the line  
 or a point are obtained by  
 striking out successively each  
 letter from the name of the  
 point. Now this  $C_n^v$  of Cap  
 orali is evidently a  $C_{n,r}^v$  or  
 $\Gamma_{n,r}^v$  where  $r=2$ . For in-  
 stance, Caporali's  $C_5^3$ , which  
 is the Desargues configuration,  
 is a  $C_{5,2}^3$  or a  $\Gamma_{5,2}^3$ , i.e., it  
 may be regarded as a sec-  
 tion of a complete 5-point  
 in space, or the projection  
 of a complete 5-plane in  
 space. Figure 1. shows it  
 lettered in black as a  $C_{5,2}^3$   
 and in red as a  $\Gamma_{5,2}^3$ .

(Since I shall confine  
 my attention for the present



to plane configuration, where  $r=2$ , I shall write  $C_n^v$  and  $\Gamma_n^v$  for  $C_{n,2}^v$  and  $\Gamma_{n,2}^v$ , the  $v$  being understood. (Also notice that here  $r=2$ ,  $p=m-v+1$ .)

Let us return here for reference that (in the plane)

- a  $C_n^{r,0} \equiv \Gamma_n^{r+1}$  is merely a point,
- a  $C_n^{r,r+1} \equiv \Gamma_n^{r,0}$  is " " line,
- a  $C_n^{r,1} \equiv \Gamma_n^{r,n}$  " " points on a line,
- a  $C_n^{r,m} \equiv \Gamma_n^{r,1}$  " " lines on a point,
- a  $C_n^{r,2} \equiv \Gamma_n^{r,n-1}$  " a complete line,
- a  $C_n^{r,n-1} \equiv \Gamma_n^{r,2}$  " " " point,
- and a  $C_n^{r,n+s} \equiv \Gamma_n^{r,s}$  } have no meaning
- and a  $C_n^{r,s} \equiv \Gamma_n^{r,n+s}$  } for  $s > 1$ .

Suppose we have given any  $C_n^v$  in a plane, and a copoint in  $\mathcal{P}_v$  from which the  $C_n^v$  is obtained by pro-



arising from a certain  $S_{n-1}$   
 upon the plane. Let us  
 separate one of these  $n$   
 points, say the one lettered  
 $a$ , from the rest. The re-  
 maining  $n-1$  points give  
 in the plane a  $C_{n-1}^v$  which  
 is that part of the  $C_n^v$  made  
 up of elements not contain-  
 ing the letter  $a$ . The re-  
 sidual  $b, c$ , etc. cut the  
 point  $a$  in  $n-1$   $S_{n-2}$ 's, giv-  
 ing us a complete figure  
 of  $n-1$   $S_{n-2}$ 's in an  $S_{n-1}$ . A  
 simple projection from a  
 point upon an  $C_{n-1}^v$  will  
 merely send this into a  
 most complete figure of  
 $n-1$   $S_{n-2}$ 's in an  $S_{n-1}$ , and



if we then project from  $S_{v-1}$  upon the plane, we obtain a  $C_{v-1}^{v-1}$ . We make these two projections in such a way that they are entirely equivalent to the single projection from the  $S_{v-2}$  upon the plane. Hence the  $C_{v-1}^{v-1}$  is that part of the  $C_v^v$  which is made up of elements containing the letter  $a$ . Moreover, any  $v-1$  of the points  $b, c, \text{etc.}$ , determine a line in  $S_v$  which projects into a line of the  $C_{v-1}^{v-1}$ . But this line in  $S_v$  into  $a$  in  $v$  points which give in the plane a point of the  $C_{v-1}^{v-1}$ . Hence the points of the  $C_{v-1}^{v-1}$ .



lies on the line of the  $C_n^v$ .

We have then Copeland's theorem:-

If in a  $C_n^v$  we separate those elements containing a given letter from those elements which do not contain it, the former form a  $C_{n-1}^{v-1}$  and the latter a  $C_{n-1}^{v-1}$  and the line of the  $C_n^v$  passes through the points of the  $C_{n-1}^{v-1}$ .

If now we regard this same configuration as a  $\Gamma_n^M$ , we can show, by a dual argument, that with respect to a given letter a  $\Gamma_n^M$  breaks up into a  $\Gamma_{n-1}^{M-1}$  containing the letter, and a  $\Gamma_{n-1}^M$  not containing it.



the elements of the  $T_{n-1}^M$  containing a given letter are the same as the elements of the  $C_{n-1}^V$  not containing it, the  $T_{n-1}^{M-1}$  and  $T_{n-1}^M$  are the same respectively as the  $C_{n-1}^V$  and  $C_{n-1}^{V-1}$  of the preceding paragraph.

Following Copson, I shall call  $C_{n-1}^V$  and  $C_{n-1}^{V-1}$  complementary when they combine to form a  $C_n^V$ . Similarly for a  $T_{n-1}^{M-1}$  and  $T_{n-1}^M$ .

Consider now two, say  $a$  and  $b$ , of the  $n$  points in  $\mathcal{D}$  which give the  $C_n^V$ . The  $n-2$  remaining points,  $c, d$ , etc., give on the plane a  $C_{n-2}^V$  made up of them.



elements of the  $C_v^v$  containing  
neither  $a$  nor  $b$ . The reports  
 $c, d, \text{ etc.}$ , cut the reports  
 $a$  in  $v-2$   $S_{v-2}$ 's, giving in  
 the plane a  $C_{v-2}^{v-1}$  made up  
 of elements of the  $C_v^v$  contain-  
ing  $a$  but not containing  
 $b$ ; and we have similarly  
 a  $C_{v-2}^{v-1}$  containing  $b$  but not  
 $a$ . The reports  $a$  and  
 $b$  intersect each other in  
 an  $S_{v-2}$ , and this is cut by  
 $c, d, \text{ etc.}$ , in  $v-2$   $S_{v-2}$ 's. The  
 simple projection and this  
 into  $v-2$   $S_{v-2}$ 's in a  
 unit  $S_{v-2}$ , and projecting  
 them for the plane, we ob-  
 tain a  $C_{v-2}^{v-2}$ , made up of  
 elements of the  $C_v^v$  containing







that the two  $\Gamma_{n-2}^{M-1}$ 's are each complementary to the  $\Gamma_{n-2}^M$ . But the  $\Gamma_{n-1}^M$ , two  $\Gamma_{n-2}^{M-1}$ 's, and  $\Gamma_{n-2}^{M-2}$  of this paragraph are respectively the same as the  $C_{n-2}^{V-2}$ , two  $C_{n-2}^{V-1}$ 's, and  $C_{n-2}^V$  of the preceding paragraph. Hence the two  $C_{n-2}^{V-1}$ 's are complementary to the  $C_{n-2}^V$ .

Conversely, if two  $C_{n-2}^{V-1}$ 's are each complementary to a  $C_{n-2}^V$ , it follows that the  $C_{n-2}^V$  may be obtained from  $n-2$  copoints in  $S_V$  and that each of the  $C_{n-2}^{V-1}$ 's may be obtained from the intersections of these  $n-2$  copoints with another copoint. Thus two latter copoints would intersect in



in  $C_{n-1}$ , and the intersection  
 of the two separates with this  
 $C_{n-1}$  would give in the plane  
 a  $C_{n-2}^{v-2}$  with which each of  
 the two  $C_{n-2}^{v-1}$ 's would be com-  
 plementary. Hence, if two  
 $C_{n-2}^{v-1}$ 's are each complementary  
 to a  $C_{n-2}^v$ , they are also com-  
 plementary to a  $C_{n-2}^{v-2}$ , and  
 the four configurations togeth-  
 er form a  $C_n^v$ . By stating  
 the same theorem for  $P$ 's and  
 then changing to  $C$ 's we have  
 that if two  $C_{n-1}^{v-1}$ 's are each  
 complementary to a  $C_{n-2}^{v-2}$ , they  
 are also complementary to a  
 $C_{n-1}^v$ , etc.

By the same method  
 used in proving the above



theorem of Copeland, we may  
 prove the following general  
 theorem which Copeland does  
 not give -

With respect to  $s$  letters,  
 say  $a, b, c, \dots, K, m, n$  ( $s \leq n$ ),  $s$   
 $C_n^v$  breaks up into  
 $s C_{n-s}^v$  of elements containing one of the letters  $a, b, c, \dots, K, m, n$ ;  
 $s C_{n-s}^v$   $\left\{ \begin{array}{l} \text{one of " " a but not b, c, } \dots, K, m, n, \\ \text{" " " " b " " a, c, } \dots, K, m, n, \\ \vdots \\ \text{" " " " n " " a, b, } \dots, K, m; \\ \text{" " " " a, b, but not c } \dots, K, m, n, \\ \text{" " " " b, c, " " a } \dots, K, m, n, \\ \vdots \\ \text{" " " " m, n " " a, b, } \dots, K; \end{array} \right.$   
 $(s) C_{n-1}^{v-2} \left\{ \begin{array}{l} \text{" " " " a, b, but not c } \dots, K, m, n, \\ \text{" " " " b, c, " " a } \dots, K, m, n, \\ \vdots \\ \text{" " " " m, n " " a, b, } \dots, K; \end{array} \right.$











every element of the latter are contained among the  $r$  letters appearing in every element of the former. For instance, the  $C_{n-1}^{r-1}$  whose elements contain  $a$  but not  $c, d, \dots, n, n$  is complementary to the  $C_{n-1}^{r-2}$  whose elements contain  $a, c$ , but not  $b, \dots, n, n$ ; but it is not complementary to the  $C_{n-1}^{r-2}$  whose elements contain  $m, n$ , but not  $a, b, \dots, k$ . All the  $C_{n-1}^{r-1}$ 's are complementary to the  $C_{n-1}^r$ , and all the  $C_{n-1}^{r-1}$ 's are complementary to the  $C_{n-1}^{r-2}$ .

Figure 2. shows a  $C_2^3$  broken up with respect to the three letters  $a, b, c$ , into a  $C_2^3$  (black), three  $C_2^2$ 's (red, green, and blue), three  $C_2^1$ 's (red, green, and



blue dotted), and a  $C_3^0$  (black dotted).  
 The red  $C_3^0$  is complementary to  
 the green and blue  $C_3^0$ 's, but  
 not to the red  $C_3^0$ 's, etc.

Suppose, conversely, that  $s$   
 $C_{n-s}^{v-1}$ 's are complementary to a  
 $C_{n-s}^{v-1}$ . The  $C_{n-s}^{v-1}$  is the projec-  
 tion of the complete figure  
 of  $n-s$  copoints in  $S_v$ , while  
 each of the  $C_{n-s}^{v-1}$ 's is the pro-  
 jection of the intersection of  
 these  $n-s$  copoints with one  
 of the  $s$  copoints  $a, b, \dots, m, n$ .  
 Any two of these  $s$  copoints,  
 say  $a$  and  $b$ , intersect in an  
 $S_{v-2}$ , and this is cut by the  
 $n-s$  copoints in  $n-s$   $S_{v-2}$ 's,  
 giving in the plane a  $C_{n-s}^{v-2}$   
 complementary to the two  $C_{n-s}^{v-1}$ 's.



Taking three of the  $s$  points, say  $a, b, \text{ and } c$ , the  $S_{v-2}$  determined by  $a$  and  $b$  is cut by  $c$  in an  $S_{v-3}$ , and this is cut by the  $v-3$  points in an  $S_{v-4}$ 's, giving in the plane a  $C_{n-3}^{v-3}$  which is complementary to the  $C_{n-2}^{v-2}$  arising from  $a$  and  $b$ . From the symmetry, it is equally well complementary to the  $C_{n-2}^{v-2}$  arising from  $b$  and  $c$ , and from  $a$  and  $c$ . Continuing in this manner, we derive the following theorem: -

If  $s$   $C_{n-1}^{v-1}$ 's are complementary to a  $C_{n-1}^v$ , the corresponding lines of any two of the  $C_{n-1}^{v-1}$ 's meet in that point of a  $C_{n-2}^{v-2}$ ; three of the  $C_{n-1}^{v-1}$ 's



(taken two at a time) determine three such  $C_{n-3}^{v-2}$ 's whose corresponding lines meet in the points of a  $C_{n-3}^{v-3}$ ; four of the  $C_{n-3}^{v-1}$ 's determine four such  $C_{n-3}^{v-3}$ 's whose corresponding lines meet in the points of a  $C_{n-3}^{v-4}$ ; - - - - - and finally, the 5  $C_{n-3}^{v-1}$ 's determine 5  $C_{n-3}^{v-3+1}$ 's whose corresponding lines meet in the points of a  $C_{n-3}^{v-2}$ ; and the entire figure is a  $C_n^v$ .

This theorem, when we make  $v = n-3$ , becomes Thomsen's theorem of perspective "pyramids"<sup>1</sup> applied to the plane.

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<sup>1</sup> Math. Ann., Vol. 19, p. 111; also Schlegel's des Geometrie, p. 614.



The figure of the  $2^2_3$  (figure 2) illustrates this theorem. It shows three  $C_3^2$ 's complementary to a  $C_3^3$ .

By permuting the above theorem for  $\Gamma$ 's and then changing to  $C$ 's we have: -

If  $s$   $C_{n-5}^{V-5+1}$ 's are complementary to a  $C_{n-5}^{V-5}$ , corresponding points of any two of the  $C_{n-5}^{V-5+1}$ 's lie on the line of a  $C_{n-5}^{V-5+2}$ ; three of the  $C_{n-5}^{V-5+1}$ 's determine three such  $C_{n-5}^{V-5+2}$ 's whose corresponding points lie on the line of a  $C_{n-5}^{V-5+1}$ ; - - - and finally the  $s$   $C_{n-5}^{V-5+1}$ 's determine  $s$   $C_{n-5}^{V-5+1}$ 's whose corresponding points lie on the line of a  $C_{n-5}^{V-5}$ , and the



the figure is a  $C_n^V$

If we express the breaking up of a  $C_n^V$  with respect to  $s$  letters thus: -

$$(1.) C_n^V \equiv C_{n-s}^V + s C_{n-s}^{V-1} + \binom{s}{2} C_{n-s}^{V-2} \dots + s C_{n-s+1}^{V-s+1} + C_{n-s}^{V-s}$$

the converse theorem just given may be roughly expressed by saying that if the first  $(s-1)$  or last two terms of the right hand member of (1.) are given, the entire expression is then determined.

If we select  $s$  letters out of the  $n$  letters of a  $C_n^V$ , and then select  $j$  letters out of the  $s$ , it follows, from the theorem given, that the elements of the  $C_n^V$  which con-



tain the  $p$  letters but do not contain the remaining  $s-p$  of the  $s$  letters form a  $C_{n-1}^{v-p}$ . Hence the number of  $C_{n-1}^{v-p}$  contained in a  $C_n^v$  is  $\binom{v}{s-p}\binom{s-p}{p}$ . This formula is entirely equivalent to, but simpler than, the one given by Caporali, viz.,  $\binom{v}{p}\binom{v-1}{s-p}$ . The same formula holds for  $T$ 's.

Consider now the problem of constructing a  $C_n^{v-1}$  complementary to a given  $C_n^v$ . The two together, as we have seen, must form a  $C_{n+1}^v$ , which is derived from  $n+1$  copoints in  $S_v$ . The  $C_n^v$  is derived from  $n$  of these copoints, and the  $C_n^{v-1}$  from the intersections of these  $n$   $C_n^v$



points with any  $(n+1)$ th. Given the  $C_n^v$  then, a complementary  $C_n^{v'}$  will be determined as soon as we choose an  $(n+1)$ th copoint in  $S_v$ . Now a copoint, or  $S_{v+1}$ , in  $S_v$  is determined by  $v$  points, provided these  $v$  points do not lie in an  $S_{v-2}$ .

The complete  $v$ -copoint in  $S_v$  has  $\binom{n}{v}$  points, through each of which pass  $v$  copoints intersecting in  $v$  lines through the point. Or, stating it differently, through each point pass  $v$  lines which determine,  $v-1$  at a time, the  $v$  copoints through the point. Evidently all  $v$  of these lines can not lie in a copoint, for this would mean



that the  $v$  copoints through the point coincide. This being the case, if we choose arbitrarily  $v$  points, one on each of these  $v$  lines, these  $v$  points cannot lie in or on  $S_{v-2}$ : for if they did, this  $S_{v-2}$  together with the point through which all the lines pass would determine an  $S_{v-1}$ , or a point, in which all  $v$  lines would lie; and we have just shown that these  $v$  lines cannot lie in or on  $S_{v-1}$ . Therefore  $v$  points chosen arbitrarily, one on each of the  $v$  lines passing through any point of the complete  $v$ -copoint in  $S_v$ , determine an  $(v+1)$ th copoint, and the intersection of the  $v$



copoints with the  $(uv)$ th will give in the plane a  $C^{v+1}$  complementary to the  $C^u$ .

Draw the  $v$  line through any point of the complete  $v$ th point in  $S_v$  project into the  $v$  line through a point of the  $C^u$  in the plane, and the  $v$  points which we choose arbitrarily on these lines project into  $v$  points of the complementary  $C^{v+1}$ . Since there  $v$  points can be chosen arbitrarily on the  $v$  line in  $S_v$ , they can be chosen arbitrarily on the  $v$  lines in the plane. Hence we have Copoints  $C^{v+1}$  thus:-

Given a  $C^u$ , it is possible to construct a  $C^{v+1}$  complementary



...line to it. We may take  
 $v$  points of the  $C_n^{v-1}$  arbitrarily  
 on the  $v$  line passing through  
 any point of the  $C_n^v$ , and the  
 rest of the  $C_n^{v-1}$  is then deter-  
 mined.

By a similar proof to that  
 just given, we can prove a  
 similar theorem for the  $T$ 's; and  
 then changing to  $C$ 's we have  
 another of Caporali's theorems:—

There are  $s^{n-v+1}$   $C_n^{v+1}$ 's com-  
 plementary to a given  $C_n^v$ . We  
 may take  $n-v+1$  lines of the  $C_n^{v+1}$   
 arbitrarily on the  $n-v+1$  points  
 lying on any line of the  $C_n^v$ ,  
 and the  $C_n^{v+1}$  is then determined.

We have shown that if  $s$   
 $C_{n-s}^{v-s+1}$ 's are complementary to a



$C_{n-v-2}^{v-2}$ , a  $C_n^v$  is thereby determined.

For  $S=v-2$ , this says that  $v-2$   $C_{n-v+2}^3$ 's complementary to a  $C_{n-v+2}^2$  determine a  $C_n^v$ . A  $C_{n-v+2}^2$  is a complete  $(n-v+2)$ -line depending upon  $2(n-v+2)$  arbitrary constants.

To construct a  $C_{n-v+2}^3$  complementary to a  $C_{n-v+2}^2$ , we draw  $n-v+1$  lines of the  $C_{n-v+2}^2$  through  $n-v+1$  points of the  $C_{n-v+2}^2$ , and this involves  $n-v+1$  arbitrary constants. Since a  $C_n^v$  is determined when  $v-2$  such  $C_{n-v+2}^3$ 's are drawn, it follows that a  $C_n^v$  is determined by

$$2(n-v+2) + (v-2)(n-v+1)$$

$$\text{or } nv - (v+1)(v-2)$$

arbitrary constants.

We see from the preceding



paragraph that a certain  $n-v+2$  lines of a  $C_n^v$  have been chosen entirely at random, and then certain  $\sqrt{v-1}$  other  $(v-2)(n-v+1)$  lines with one degree of freedom, the rest of the  $C_n^v$  is entirely determined. Hence Caporali's theorem: -

In order that a  $C_n^v$  may be complementary to a  $C_n^{v-1}$ , it is sufficient that  $(v-1)(n-v+1) + 1$  ( $= n-v+2 + (v-2)(n-v+1)$ ) of its lines, properly chosen, pass through the corresponding points of the  $C_n^{v-1}$ .

Let there be given a  $C_n^{v+1}$  and a  $C_n^{v-1}$  in a plane. The  $C_n^{v+1}$  and  $C_n^{v-1}$  are derived respectively from a  $S_4$ : -



$\wedge^{n-1} S_{v+1}$  and  $n$   $S_{v-2}$ 's in  $\wedge^{n-1} S_{v+1}$ ; and we may suppose the  $S_{v-1}$  to be in the  $S_{v+1}$ .<sup>1</sup> Now in general the  $n$   $S_v$ 's will not cut the  $S_{v-1}$  in the  $n$   $S_{v-2}$ 's. If they do, passing any  $S_v$  through the  $S_{v-1}$ , the intersections of the  $n$   $S_v$ 's with this  $(n+1)$ th  $S_v$  gives in the plane a  $C_n^v$  complementary to both the  $C_n^{v+1}$  and  $C_n^{v-1}$ . There are a single infinity of  $S_v$ 's.

<sup>1</sup> The theorem breaks down for the simple case of  $v=1$ , or the dual case,  $v=n$ ; i.e., when  $v=1$  or  $n$ , there are always a single infinity of  $C_n^v$ 's complementary to any  $C_n^{v-1}$  and to any  $C_n^{v+1}$ .



through in  $S_{n-1}$  in  $S_{n+1}$ , and  
therefore we have Liporali's  
theorem: -

Given a  $C^{n+1}$  and a  $C^{n-1}$ ,  
in general there is no  $C^n$   
complementary to both. If  
there is one, there are a sin-  
gle infinity of them. One  
obtains that a simple pen-  
cil of  $C^n$ 's, characterized by  
the fact that their lines in-  
tersect about fixed points while  
their points lie along fixed  
lines.

For example, one can not,  
in general, draw a  $C_3^4$  com-  
plementary to a given  $C_3^3$  and  
a given  $C_3^1$ . If however the  
three lines through a point are



and three points on a line are such that one triangle can be drawn inscribed to the former and circumscribed to the latter, then a singly infinite number of such triangles can be drawn.

## § 2.

We pass now to the consideration of configurations in the ordinary three-dimensional space.

A  $C_{n,3}^V$  or  $\Gamma_{n,3}^M$  is a configuration in space included under the general definition of a  $C_{n,r}^V$  or  $\Gamma_{n,r}^M$ . (See



I shall confine my attention in this section to space configurations where  $r=3$ , I shall write  $C_n^v$  and  $T_n^v$  for  $C_{n,3}^v$  and  $T_{n,3}^v$ , the 3 being understood. Also, when  $r=3$ ,  $\mu = n-v+2$ .)

A  $C_n^v$  consists of  $\binom{n}{v}$  points,  $\binom{n}{v-1}$  lines, and  $\binom{n}{v-2}$  planes. There are  $\binom{n-v+2}{2}$  and  $n-v+1$  points respectively on each plane and line;  $\binom{v}{v-2}$  and  $v-1$  planes respectively on each point and line; and  $v$  and  $n-v+2$  lines respectively on each point and plane.

A  $T_n^\mu$  is the same as a  $C_n^v$ , and a  $T_n^v$  is the dual of the  $C_n^v$ . A  $C_n^v$  or  $T_n^v$  is self-dual if  $\mu=v$ , i.e., if  $3v=n+2$ . The points, lines, and planes of



$C_n^v$  are named respectively by combinations of  $v, v-1, \dots$  and the letters out of  $n$ ; while the points, lines, and planes of a  $T_n^m$  are named respectively by combinations of  $m-2, m-1$ , and  $m$  letters out of  $n$ .

For reference, we note that (in space)

$C_n^0 \equiv T_n^{n+2}$  is merely a point,  
 $C_n^{n+2} \equiv T_n^0$  " " " plane,  
 $C_n^{n+1} \equiv T_n^{n+1}$  " " "  $n$  points on a line,  
 $C_n^{n+1} \equiv T_n^1$  " " " planes " " " ,  
 $C_n^{n+2} \equiv T_n^{n+2}$  " " " a complete  $n$ -line on a plane,  
 $C_n^n \equiv T_n^2$  " " " " " " " point,  
 $C_n^{n-1} \equiv T_n^{n-1}$  " " " " " " "  $n$ -plane,  
 $C_n^{n-1} \equiv T_n^3$  " " " " " " "  $n$  point,  
 and  $C_n^{n+3} \equiv T_n^{n+3}$  } have no meaning  
 and  $C_n^{2-3} \equiv T_n^{n+3}$  } for  $s \geq 2$ .



I shall merely state the theorem for the space configuration without proof, since the proofs are entirely analogous to the proofs of the theorem for the plane.

The theorem for the breaking up of the configuration with respect to any  $s$  letter may be expressed by the same general formula used for the plane configuration, viz.,

$$C_{n-s}^V \equiv C_{n-s}^V + s C_{n-s-1}^{V-1} + \binom{s}{2} C_{n-s-2}^{V-2} + \dots + s C_{n-s-1}^{V-s+1} + C_{n-s-1}^{V-s}$$

but this expression has no extended meaning for space configurations.

When  $s=1$ , we have that with respect to a single letter, say  $a$ ,



a  $C_n^V$  breaks up into a  $C_{n-1}^{V-1}$  whose elements enter  $a$ , and a  $C_{n-1}^V$  whose elements do not enter  $a$ . The planes and lines of the  $C_{n-1}^{V-1}$  are incident respectively with the lines and points of the  $C_{n-1}^{V-1}$ . As in the plane, I shall call a  $C_{n-1}^V$  and  $C_{n-1}^{V-1}$  complementary when they are so related.

With respect to two letters, say  $a$  and  $b$ , the  $C_n^V$  breaks up into a  $C_{n-2}^V$ , two  $C_{n-2}^{V-1}$ 's, and a  $C_{n-2}^{V-2}$ . The  $C_{n-2}^{V-1}$ 's are both complementary to the  $C_{n-2}^V$  and also to the  $C_{n-2}^{V-2}$ . The planes of the  $C_{n-2}^V$  are incident with the points of the  $C_{n-2}^{V-2}$ . It will be convenient to speak of



a  $C_{v1}$  and  $C_{v2}$  are related as supplementary. Evidently if a  $C_{v1}$  is supplementary to both a  $C_{v1}$  and a  $C_{v2}$ , that two latter configurations are supplementary.

If a  $C_{v1}$  and  $C_{v2}$  are supplementary, there is a single infinity of  $C_{v1}$ 's complementary to both. This gives us a pencil of  $C_{v1}$ 's, characterized by the fact that their planes are all about fixed line while their points are along fixed line. If we know a line of the  $C_{v1}$  in a plane of the  $C_{v2}$  and through the point of the  $C_{v2}$  in that plane, the  $C_{v1}$  is then completely determined. Any two such  $C_{v1}$ 's together



with the  $C_{v-1}^{n-1}$  and  $C_{v-1}^{n-2}$  make up a  $C_v^n$ .

As a very simple illustration of this, take the case where  $v=3$  and  $n=4$ . We have then a  $C_2^3$  supplementary to a  $C_1^4$  (i.e., two planes through a line, and two points on a line), the two points of the latter incident with the two planes of the former. A  $C_1^4$  complementary to both of them means two lines in a plane, each of the two lines incident with a plane of the  $C_2^3$  and with a point of the  $C_1^4$ . Evidently such a  $C_1^4$  is determined as soon as one of its lines is properly drawn. Two such  $C_1^4$ 's together with the  $C_2^3$  and  $C_1^4$  form a  $C_4^4$ , or tetrahedron.





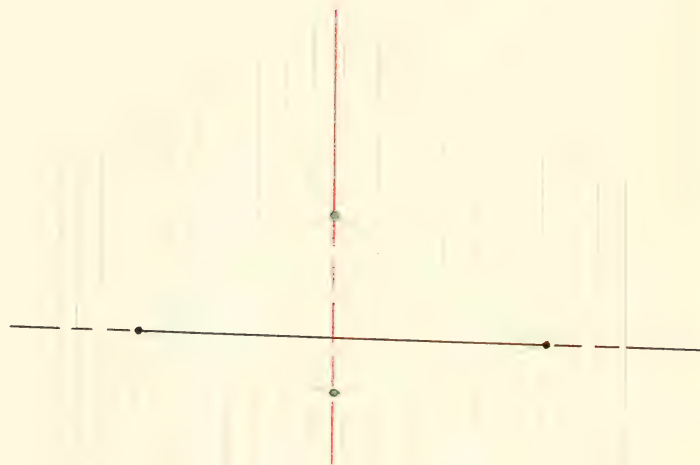


Figure 3.  
(to face page 54.)

hedron. Figure 3. shows the  $C_2^2$  in red, the  $C_2^1$  in black, and the two  $C_2^0$ 's in green.

As I have already said, the breaking up of a configuration with respect to  $s$  letters may be expressed by the formula

$$C_n^{iv} \equiv C_{n-s}^{iv} + s C_{n-s}^{iv-1} + \frac{s(s-1)}{2} C_{n-s}^{iv-2} - \dots + s C_{n-s}^{iv-s+1} + C_{n-s}^{iv-s}$$

Any  $C_{n-s}^{iv-p}$  of this expression is a configuration made up of those elements of the  $C_n^{iv}$  which contain a certain  $p$  of the  $s$  letters, but not the remaining  $s-p$ . A  $C_{n-s}^{iv-p}$  and a  $C_{n-s}^{iv-p+1}$  are complementary if the  $p-1$  letters distinguishing the latter are contained among the  $p$  letters



distinguishing the former. A  $C_{n-1}^{v-1}$  and a  $C_{n-1}^{v-1+p+2}$  are supplementary if the  $p-2$  letters of the latter are contained among the  $p$  letters of the former.

There are  $\binom{v}{p} \binom{v}{p} C_{n-1}^{v-1}$  contained in a  $C_n^v$ . For example the number of  $C_3^1$ 's (triangles) contained in a  $C_4^3$  (tetrahedron) is  $\binom{4}{3} \binom{4}{3} = 4$ .

If  $s$   $C_{n-1}^{v-1}$ 's are supplementary to a  $C_n^v$ , the corresponding lines of any two of the  $C_{n-1}^{v-1}$ 's meet in the points of a  $C_{n-2}^{v-2}$ ; three of the  $C_{n-1}^{v-1}$ 's determine three such  $C_{n-2}^{v-2}$ 's whose corresponding lines meet in the points of a  $C_{n-3}^{v-3}$ ; --- and finally, the  $s$   $C_{n-1}^{v-1}$ 's determine  $s$   $C_{n-2}^{v-2}$ 's.



whose corresponding lines meet  
in the point of a  $C_{n-1}^{v-1}$ ; and  
the entire configuration is a  
 $C_n^v$ .

As a simple example of  
this theorem consider three  $C_4^2$ 's  
(complete 4-lines in a plane)  
complementary to a  $C_4^3$  (tetrahedron).  
A  $C_4^2$  complementary to a  $C_4^3$   
is merely a planar section of  
the  $C_4^3$ . Evidently, then, the cor-  
responding lines of any two  
of the  $C_4^2$ 's meet in the point  
of a  $C_4^1$  (four points on a line)  
and the lines of the  $C_4^1$ 's meet  
in the point of a  $C_4^0$  (which is  
merely a point).

(When  $v = n-1$ , the theorem  
just given is the three-line



sional case of Veronese's perspective "pyramid" theorem.<sup>2</sup>

There is also the dual theorem to the one just stated.

Given a  $C_n^v$ , there are  $\infty^v$   $C_n^{v-1}$ 's complementary to it. To construct such a  $C_n^{v-1}$ , we may take  $v$  of its points arbitrarily on the  $v$  lines passing through any point of the  $C_n^v$ . The entire  $C_n^{v-1}$  is then determined.

There are  $\infty^{n-v+2}$   $C_n^{v+1}$ 's complementary to a given  $C_n^v$ . We may take  $n-v+2$  of its planes arbitrarily passing through the  $n-v+2$  lines lying

<sup>2</sup> Math. Ann., Vol. 19, p. 171; also Grundzüge der Geometrie, p. 614.



in any plane of the  $C_n^v$ , and the  $C_n^{v+1}$  is then determined.

To construct a  $C_n^{v-2}$  supplementary to a given  $C_n^v$ , we may take  $v-1$  of its points arbitrarily on the  $v-1$  planes passing through any line of the  $C_n^v$ . There are then  $s^{v-1}$  such  $C_n^{v-2}$ 's.

There are  $s^{2(n-v+1)} C_n^{v+2}$ 's supplementary to a given  $C_n^v$ . We may take  $n-v+1$  of the planes of the  $C_n^{v+2}$  arbitrarily passing through the  $n-v+1$  points lying on any line of the  $C_n^v$ , and the  $C_n^{v+2}$  is then determined.

A  $C_n^v$  is determined by  $3(n-v+3) + (v-3)(n-v+2)$  or  $nv - (v+1)(v-3)$  constants.



Given a  $C_n^{v-2}$  and a  $C_m^{v-2}$ ,  
 there is not, in general, any  
 $C_n^{v-2}$  supplementary to the for-  
 mer and complementary to  
 the latter; nor is there, in  
 general, any  $C_n^{v-1}$  complementary  
 to the former and supplementary  
 to the latter. If, however, there  
 is one such  $C_n^{v-2}$  or one such  
 $C_n^{v-1}$ , there are  $2^2$  of both of  
 them. If we have given a  $C_n^{v-2}$   
 and a  $C_m^{v-3}$  such that there  
 are  $C_n^{v-1}$ 's complementary to the  
 former and supplementary to  
 the latter, we can take our  
 plane of the  $C_n^{v-1}$  arbitrarily  
 through any point of the  $C_m^{v-3}$ ,  
 and the  $C_n^{v-1}$  is then determined.  
 Similarly, a  $C_n^{v-2}$  supplementary



to the  $C_n^{v-1}$  and complementary to the  $C_n^{v-2}$  is entirely determined when one of its points has been chosen arbitrarily on a plane of the  $C_n^v$ .

Given a  $C_n^{v+1}$  and a  $C_n^{v-2}$ , there is not, in general,<sup>1</sup> any

<sup>1</sup> When  $v=2$  or  $n$ , there are always  $10^5$   $C_n^{v-1}$ 's supplementary to any given  $C_n^v$  and any given  $C_n^{v-2}$ . Also, there are always  $10^4$   $C_4^3$ 's supplementary to any  $C_4^5$  and any  $C_4^1$ , i.e., given four planes through a line and four points on a line, we can always construct a tetrahedron with its vertices on the four planes, and its planes through the four points. Two sections of this tetrahedron are



$C_n^v$  supplementary to both of them.  
 If, however, there is one such  
 $C_n^v$ , there are  $\infty^4$  of them. If  
 we have given a  $C_{n+2}^{v+2}$  and a  
 $C_n^{v-2}$  of such a kind that it  
 is possible to construct a  
 $C_n^v$  supplementary to both, we  
 can take two points of the  $C_n^v$   
 on any two planes of the  $C_{n+2}^{v+2}$ ,  
 or two planes of the  $C_n^v$  on any  
 two points of the  $C_{n+2}^{v+2}$ , and  
 the  $C_n^v$  is then entirely de-  
 termined. The  $\infty^4$  such  $C_n^v$   
 form a sheaf, characterized

by the fact that it can be  
 taken arbitrarily on any two  
 of the four planes, or two points  
 may be taken arbitrarily through  
 any two of the four points.



by the fact that their plane  
involve about fixed points,  
while their points move over  
fixed planes.

§ 3.

It is well known that the  
Sawgrass configuration in the  
plane, the  $\Gamma_{5,2}^3$ , (I shall regard  
it for my present purposes  
as a  $\Gamma_{5,2}^3$  rather than a  $C_{5,2}^3$ )  
determines a certain cone  
with respect to which it is  
self-polar. Veronese shows<sup>1</sup>, more  
generally, that the complete fig.

<sup>1</sup> Math. Ann., Vol. 17, p. 194.



one of two properties "fundamental pyramids" in  $S_r$ , i.e., the configuration  $T_{r+3, r}^{(r+1)}$ , determine a certain  $(r-1)$ -dimensional quadric spread,  $\mathbb{I}_{r-1}^*$ , with respect to which it is self polar.

This general theorem may be very simply proved<sup>1</sup> as follows:

Let  $a, b, c, \dots, K$  be  $r+3$  arbitrarily chosen points in  $S_{r+1}$ , and let  $Q_1, Q_2, \dots, Q_t$  (where  $t = \frac{r^2+3r}{2}$ ) be  $t$   $(r-1)$ -dimensional

<sup>1</sup> This proof is an extension of a proof for the plane configuration given by Prof. Morley in his lecture on Geometry. Cf. Am. Bulletin, Vol. 4 (1901), p. 5.



quadrics,  $F_r^{2,1}$ , passing through the  $r+3$  points. Then

$$Q \equiv \lambda_1 Q_1 + \lambda_2 Q_2 + \dots + \lambda_t Q_t$$

represents  $\infty^{t-1}$  quadrics through the points. Now in  $S_{r+1}$  there are altogether  $\infty^{\frac{(r+1)(r+2)}{2}}$   $F_r^{2,1}$ , and therefore through any  $r+3$  points there are  $\infty^{\frac{(r+1)(r+2)}{2} - (r+3)} = \infty^{t-1}$ .

Therefore  $Q$  is any  $F_r$  through the  $r+3$  points.

If now we take the section of the complete  $(r+3)$ -point by an  $S_r$ , we obtain the configuration  $\Gamma_{r+3,r}^{r+1}$ . The sections of  $Q, Q_1, Q_2, \dots, Q_t$  respectively will be the  $(r-1)$ -dimensional quadrics,  $Q', Q'_1, Q'_2, \dots, Q'_t$ , and we will have



$$Q' \equiv \lambda_1 Q'_1 + \lambda_2 Q'_2 + \dots + \lambda_s Q'_s$$

Now in  $S_r$  there are all together  $s^t$   $F_{r-1}^2$ 's, and  $Q'$  represents  $s^{t+1}$  of them, any  $t+1$  of which are connected by a linear relation. Hence  $Q'$  represents all  $F_{r-1}^2$ 's apolar to a certain  $\mathbb{P}_{r-1}^2$ , which we shall call simply  $\mathbb{P}$ .

We have thus shown that every  $F_r^2$  through the  $r+3$  points in  $S_{r+1}$  gives in the section by the  $S_r$  an  $F_{r-1}^2$  which is apolar to  $\mathbb{P}$ . As a special case of an  $F_r^2$  through the  $r+3$  points, we may take the  $S_r$  through  $r+1$  of them, say  $c, d, \dots, k$ , and any  $S_r$  through the remaining two,  $a$  and  $b$ . The



section of this by the  $S_r$  is the  
 $S_{r-1}^{cd \dots k}$  of the  $\Gamma_{r+3, r}^{r+1}$  together with  
 any  $S_{r-1}$  through the point,  $ab$ .  
 Hence the point  $ab$  must be  
 the pole of the  $S_{r-1}^{cd \dots k}$  with  
 respect to  $\mathbb{I}$ . Hence the  
 polar system, or "polarity",  $\mathbb{I}$ ,  
 which is uniquely determined  
 when the  $\Gamma_{r+3, r}^{r+1}$  is given, sends  
 any point of the configura-  
 tion, as  $ab$ , into the  $S_{r-1}^{cd \dots k}$ .  
 Consider now the plane  
 configuration  $\Gamma_{5, 2}^{13}$ .

Cayley shows<sup>1</sup> that the  $\Gamma_{5, 2}^{13}$

---

<sup>1</sup> Crelle's Journal, Vol. 31 (1846);  
 also Collected Works, Vol 1, p. 317.

Cf. also J. T. Graves, Phil. Mag.,  
 Vol. 15 (1839), p. 121.



breaks up into two pentagons each inscribed and circumscribed to the other. Any cyclic arrangement of the five letters, as  $abcde (= bcdca = edcba, \text{ etc. })$ , represents one of these pentagons, i.e., the pentagon whose vertices are

$ab, bc, cd, de, \text{ and } ea,$

and whose sides are

$abc, bcd, cde, dea, \text{ and } eab.$

The remaining five points and five lines of the configuration form the associated pentagon,  $acebd$ . In both of these pentagons, each vertex is the pole of the opposite side with respect to the polarity  $\Pi$  of the configuration.



To say that a polarity sends each vertex of a pentagon into the opposite-side is to say that each pair of vertices not connected by a side are a pair of conjugate points with respect to the polarity. This places five linear conditions on the polarity, and hence determines it uniquely.<sup>1</sup> A polarity may then be given by a pentagon.

Of the two pentagons which

<sup>1</sup> Cf. Reye, Geometrie der Lage, (Auffl. 3) Vol. 2, p. 125. See also a more general theorem given by Kohn, Math. Ann., Vol. 46 (1895), p. 303.



from a  $\Gamma_{5,2}^1$ , one may be taken arbitrarily and a side or vertex of the second to satisfy a single condition — to pass through a given point, for instance, or lie in a given line. (A  $\Gamma_{5,2}^3$  depends upon  $3.5 - (3+1)(3-0)$  or 11 constants. Ten of these are involved in the arbitrary choice of the first pentagon, leaving one degree of freedom to the second pentagon.) But the polarity  $\mathbb{P}$  is determined as soon as the first pentagon is chosen, and hence the second pentagon is necessarily self-polar with respect to the polarity-determined by the first pentagon.





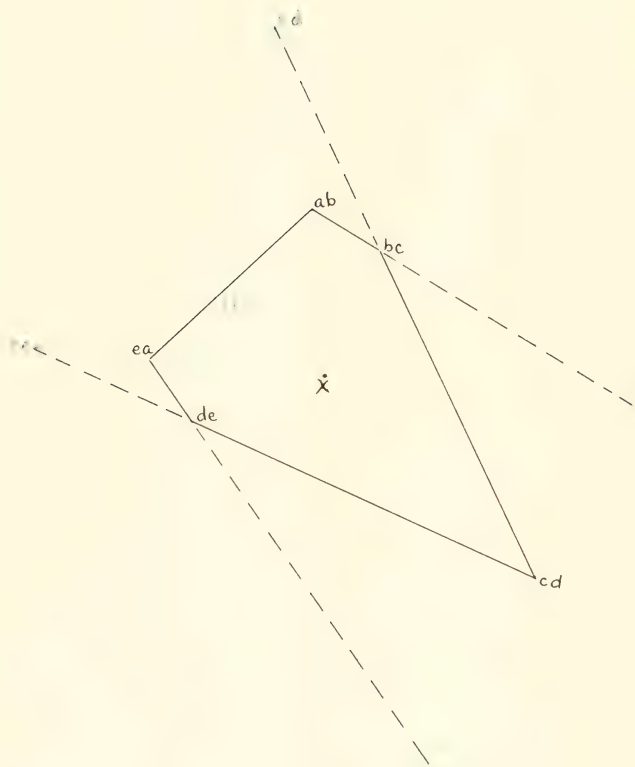


Figure 4.  
(to page 76)

This gives a method of constructing the polar of any point with respect to a polarity  $\Phi$  given by a self polar pentagon.

Consider the given pentagon as the pentagon  $abcde$  of a  $\Gamma_{5,2}^2$ . If we make any side, as  $acc$ , of a circumscribed pentagon,  $acebd$ , pass through the point,  $x$ ,  $acebd$  is then entirely determined (see figure 4.). Since  $acc$  is the polar of  $bd$ , the polar of  $x$  must pass through  $bd$ .

For construction purposes it is not necessary to draw the entire pentagon  $acebd$ , but only three sides of  $bd$ , i.e.



sufficient to determine the center  
b.d. The construction can be  
conveniently remembered as  
follows: -

Number the vertices and  
sides of the pentagon suc-  
cessively, 1, 2, 3, ..., 10, assigning  
odd numbers to the vertices  
and even numbers to the  
sides. Then

join  $x$  to 1, to meet 4 at  $A$ ,  
"  $A$  " 7, " " 10 "  $B$ ,  
"  $B$  " 3, " " 6 "  $C$ ,  
and  $C$  is then on the polar of  
 $x$ . Then shift the numbers,  
replacing 3 by 1, 4 by 2, ..., 2 by 10,  
and repeat the process, ob-  
taining another point  $C'$ . Then  
 $CC'$  is the polar of  $x$ .



A dual construction given  
the pole of a given line.

Two Pentagons, representing  
two polarities  $\mathbb{I}$  and  $\mathbb{I}'$ , of  
which the first sends points  
into lines, and the second  
sends lines into points, give  
a collineation. The common  
polar triangle of  $\mathbb{I}$  and  $\mathbb{I}$   
is the fixed triangle of the  
collineation. If  $ABCDE$  is  
a pentagon giving the polar-  
ity  $\mathbb{I}$ , the triangle whose  
vertices are

$$A, B, (\overline{AB}, CT)$$

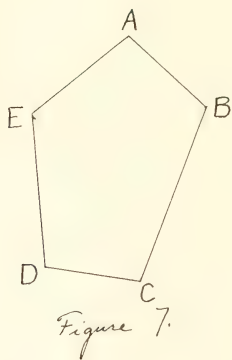
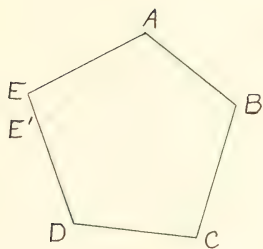
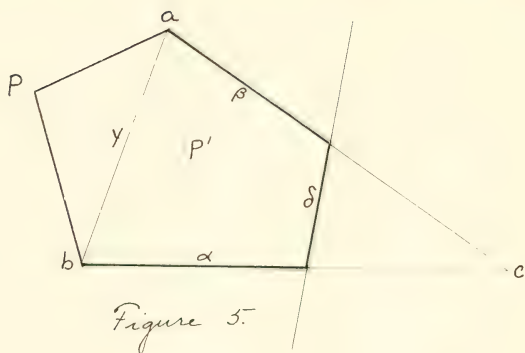
is evidently a self-polar triangle  
of  $\mathbb{I}$ <sup>1</sup>. If then a given triangle

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<sup>1</sup> See reference to Page on p. 68







(To face page 73.)

with vertices  $a, b$ , and  $c$ , and sides  $x, y$ , and  $z$ , is to be the fixed triangle of a collineation which in addition to each a given point  $P$  into a given point  $P'$ , we can at once draw two pentagons which will give this collineation. Let  $S$  be any line in the plane. The two pentagons may be

$$P, a, (b.d), (d.x), b, \text{ and } c$$

$$P', a, (b.d), (d.x), b, \text{ and } c.$$

(See figure 5.)

Two pentagons  $ABCDE$  and  $A'B'C'D'E'$ , where  $E'$  lies on  $DE$ , (see figure 6.) give a collineation in which the point  $D$  and all points on the line  $AB$  are fixed.



ed points, while  $AB$  and all the lines through  $D$  are fixed lines. This collineation sends  $E$  into  $E'$ , and it sends any point  $P$  into a point  $P'$  on the line  $PD$  such that the double ratio  $P, P'; D, AB$  is equal to the double ratio  $E, E'; L, AB$ .

Chasles calls attention<sup>1</sup> to a collineation determined by a pentagon. With the pentagon  $ABCDE$  is naturally associated the pentagon of the diagonals,  $ACEBD$ , (see figure 7) and these two

<sup>1</sup> Über die Anwendung der quadratischen Substitution auf die Verbindungen vom Grade und die geometrische Theorie der ebenen fünfseitigen, Math. Ann., Vol. 4 (1811)



pentagons give the Clebsch octahedron.

Passing now to three dimensions, we have the configuration  $\Gamma^4_{6,3}$  consisting of two perspective tetrahedra with their center and plane of perspective. The points, lines, and planes of this configuration are denoted respectively by combinations of two, three, and four letters out of six. The point  $ab$  is the pole of the plane  $cdef$ , and the lines  $abc$  and  $def$  are conjugate with respect to a certain polarity.  $\text{I}$ .

Any cyclic arrangement



7E  
Of the six letters, as  $abcdef$ ,  
represents a hexagon whose  
vertices are

$ab, bc, cd, de, ef$ , and  $fa$ ;  
whose lines are

$abc, bcd, cde, def, efa$ , and  $fab$ ;  
and whose planes are

$abcd, bcde, cdef, defa, efab$ , and  $fabc$ .  
Each vertex of this hexagon is  
the pole of the opposite plane  
with respect to  $\mathbb{E}$ .

In addition to the vertices  
of this hexagon, there are  
several other points of the con-  
figuration, namely

$ac, ce, ea, bd, df, fb$ ,  
 $ad, be$ , and  $cf$ ;

of which the first six lie on  
the lines of the hexagon, while



1.  
the last three lie one each on  
the lines of intersection of the  
three pairs of opposite planes  
of the hexagon. (For instance,  
ad lies on the line of intersec-  
tion of abcd and defa.) Sim-  
ilarly, there are nine planes  
of the configuration in addition  
to those of the hexagon,  
namely

abce, bedf, cdea, defb, efac, fabd,  
beaf, cdfc, and deab;

of which <sup>the first</sup> six pass through the  
lines of the hexagon, and the  
last three through the joins of  
the three pairs of opposite  
vertices.

From these nine points  
and nine planes, we have



pick out (in three different ways) a second hexagon, which, like the first, may be represented by a cyclic arrangement of the six letters. Such a hexagon is  $ad+bec$ . In this hexagon, as in the first, each vertex is the pole of the opposite plane with respect to  $\Phi$ . Moreover, the two hexagons,  $abcd+ef$  and  $ad+bec$ , are mutually related. Of the six vertices of each, four lie on  $\Delta$ , and two on the intersections of pairs of opposite planes of the other. Of the six planes of each, four pass through  $\Delta$ , and two through the joins of  $\Delta$



of opposite vertices of the other. Also four lines of each pass through points and lie in planes of the other, and the remaining two lines of each set three lines of the other, i.e., lie on a quadric surface determined by the other.

If each vertex of a hexagon is the pole of the opposite plane with respect to a polarity- $\Phi$ , this polarity is determined uniquely as soon as the hexagon is given.<sup>1</sup> Hence as soon as the hexagon  $abcdef$  is given, the polarity  $\Phi$  of the configuration is de-

---

<sup>1</sup> See reference to Kohn on p. 68.



terminated.<sup>2</sup>

I wish now to show that the hexagon  $abcdef$  may be taken arbitrarily, and a plane or point of the second hexagon,  $adfbce$ , to satisfy a single relation (a plane to pass through a given point, for instance),

---

<sup>2</sup> Dr. Kassar, in a paper on The Double-six Configuration, Amer. Journal of Math., Vol. 25 (1903), calls attention to a polarity,  $\Omega$ , connected with the double-six. In a letter to Prof. Morley he shows that a skew-hexagon self-polar as to  $\Omega$  (and hence determining  $\Omega$ ) may be selected in twenty different



and that the second hexagon and the entire configuration will then be determined. (The  $\Gamma_{6,3}^y$  depends upon  $4 \cdot 3 - (7+1)(4-3)$  or 19 constants. Eighteen of the

ways from the lines of the double-six. If  $L_i$  and  $M_i$  ( $i=1, 2, \dots, 6$ ) are the lines of the double-six, then any corresponding triples,  $l_1, l_2, l_3$  and  $M_1, M_2, M_3$  form such a hexagon.

In this same letter Dr. Reeser suggests a method of constructing a polarity when defined by a self polar hexagon. He reduces it to the construction of a second plane polarity defined by plane pentagon.



constants are involved in choosing the first hexagon arbitrarily, leaving the degree of freedom to the second hexagon.

Suppose  $abcdef$  to be chosen arbitrarily. We then make any plane, say  $adt$ , of the second hexagon pass through a given point  $x$  and the line  $fab$  of the first hexagon. Let  $adt$  cut  $def$  in  $dt$  and  $bed$  in  $bd$ . Then let

$bdt$ (the join of $bd$ and $dt$ )	cut $fab$ in $fb$ ;
$fbe$ ( $fb.be$ )	" $bced$ " $be$ ;
$[abe$ ( $ab.be$ )	" $efad$ " $af$ ];
$bec$ ( $be.bc$ )	" $cde$ " $ec$ ;
$eca$ ( $ec.ae$ )	" $abc$ " $ca$ ;
and $cad$ ( $ca.cd$ )	" $defa$ " $ad$ ;





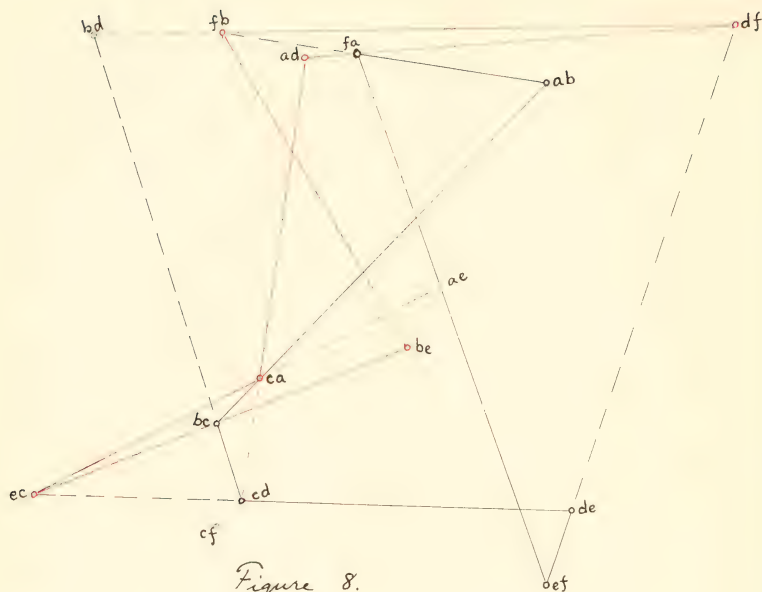


Figure 8.

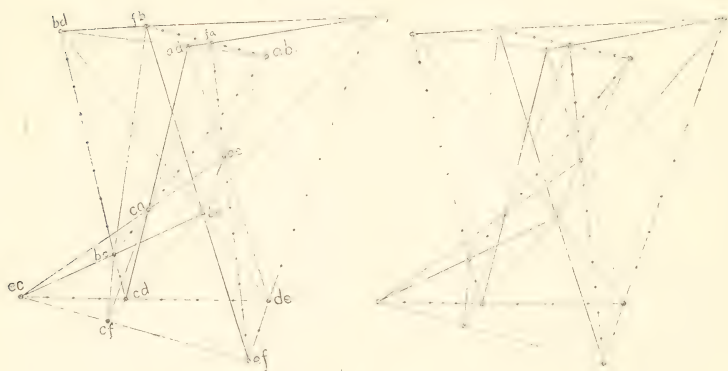


Figure 9.  
(To face page 83)

and if our hexagon is to close  
 in properly, it should lie  
 in the plane  $adtb$ . In order  
 to show that this will always  
 be the case, I shall show  
 that, by the addition of a few  
 lines to the figure already  
 constructed, we obtain the  
 complete figure of the per-  
 spective tetrahedron. (Figure 8  
 shows the hexagon  $abcdef$  in  
 black, the hexagon  $adtbec$  in  
 red, and the remaining lines  
 of the figure in green. This  
 same figure may be seen in  
 relief by looking at figure 9  
 through a stereoscope.)

Join  $ec$  and  $ef$  by the  
 line  $cef$ . Then the three lines



$fbc$  ( $fb.bc$ ),  $fac$  ( $fa.ca$ ), and  $cdf$  ( $cd.df$ ) will cut  $cet$ ; for the line respectively in the three planes  $fbee$  ( $fb.be.et.bc.ec$ ),  $face$  ( $fa.ca.ct.ca.ec$ ), and  $fede$  ( $df.de.et.cd.ec$ ) passing through  $cet$ . Moreover, these three lines all cut  $cet$  at the same point  $q$ , because each pair of them lies in a plane which does not contain  $cet$ ; viz.,  $fb$  and  $fac$  in  $fabc$  ( $fa.fb.ab.bc.ca$ );  $fac$  and  $cdf$  in  $facd$  ( $fa.df.ad.cd.ac$ ), and  $cdf$  and  $fbc$  in  $fbcd$  ( $fb.df.bd.cd.bc$ ). Also the line  $bde$  ( $be.de$ ) passes through  $bd$  since it lies in the two planes  $bdef$  ( $be.de.fb.df$ ) and  $bdec$  ( $be.de.bc.cd$ ), both of



which contain  $bd$ .

Examination of the figure now shows that  $ca, bc, cd, ct$  and  $ac, bc, dc, ct$  are two perspective tetrachords with  $ca$  as centre of perspective. Five pairs of corresponding edges meet in the five points  $bd, tb, dt, ta$ , and  $ab$  of the plane  $adtb$ . The sixth pair of edges,  $cd$  and  $dc$  meet also meet in a point of this plane. But  $cd$  lies in the plane  $dcta$ , and hence  $cd$  meets  $dcta$  in the point  $cd$  of the plane  $adtb$ . Since the points  $ta$  and  $dt$  lie in the plane  $dcta$ ,  $cd$  lies on a line with these two points.



and, drawing this line, we  
 complete the hexagon. I have  
 shown then that one  
 hexagon may be taken arbit-  
 rarily, and one plane of  
 the second through any point  
 in space, and that the sec-  
 ond hexagon is then deter-  
 mined.

This gives then the meth-  
 od of constructing the polar  
 plane of any point,  $x$ , with  
 respect to a polarity  $E$   
 given by a self-polar hexagon.  
 Regarding the given hexagon  
 as the hexagon about  $E$ , we  
 may pass a plane of a  
 second hexagon,  $ab+bc$ , through  
 the point  $x$ , and the second



hexagon is then determined, and the vertex opposite the plane passed through  $x$  will lie on the polar plane of  $x$ . The same rule constructs the entire second hexagon. The construction may be simply stated as follows:—

Let 1, 2, 3, 4, 5, 6 be the vertices of the given hexagon in order. Pass a plane through  $x$ , 1, and 2 cutting  $\overline{34}$  and  $\overline{56}$  in  $A$  and  $A'$ . Let  $\overline{AA'}$  cut  $\overline{12}$  in  $B$ , and  $\overline{B6}$  cut the plane  $\overline{345}$  in  $C$ .  $\overline{C3}$  will then cut  $\overline{45}$  in the point  $D$  which is on the polar of  $x$ . Now shift the numbers, replacing 2 by 1, 3 by 2, ... 6 by 5.



and repeat the process, obtaining thus a second point  $D'$ . Similarly for a third point  $D''$ . Then  $DD'D''$  will be the polar plane of  $\alpha$ .

A dual construction will give the polar point of a given plane.

The combination of two space polarities gives a space collineation, and hence a space collineation may be given by two hexagons.

If a given tetrahedron, with vertices  $a, b, c$ , and  $d$ , and faces  $\alpha, \beta, \gamma$ , and  $\delta$  is to be the fixed tetrahedron of a collineation, which is in addition to send a given point



$P$  into a given point  $P'$ , this collineation may be given by the two hexagons

$P, a, (y.s.e), (s.x.e), (x.s.e), b$   
and  $P', a, (y.s.e), (s.x.e), (x.s.e), b$ ;  
where  $e$  is any plane.

The two hexagons  $ABCDEF$  and  $ABCDEF'$ , where  $F'$  lies on  $EF$ , give a collineation in which the point  $E$  and all points of the plane  $ABC$  are fixed points, and the plane  $ABC$  and all planes through  $D$  are fixed planes.

For an analytic construction of the plane configuration,  $\Gamma_{5,2}$ , we may take one fixed point  $\in B_1$ .



$$z_i = 0 \quad (i=1, \dots, 5)$$

with the relation

$$\sum_{i=1}^5 z_i = 0$$

The complete 5-point in  $S_5$  consists then of these five points together with the line

$$z_i + \lambda z_j = 0 \quad (i \text{ and } j = 1, \dots, 5; i \neq j)$$

and the two planes

$$\lambda_K = 0 \quad (K, M, \text{ and } V = 1, \dots, 4; K \neq V)$$

and  $\lambda_M - \lambda_V = 0$

We cut this figure by the plane

$$\sum_{K=1}^4 \alpha_K X_K = 0$$

and take as reference triangle in this plane the sections of the plane

$$X_1 = 0, \quad X_2 = 0, \quad X_3 = 0$$

and as origin of lines the



section of the plane

$$\sum_K \lambda_K = 0$$

The two points of the  $\Gamma_{5,2}^2$  are then

$$x_j \bar{x}_i - x_i \bar{x}_j = 0 \quad (i \neq j = 1, \dots, 5; i \neq j),$$

and the two lines are

$$\lambda_K = 0 \quad (K, \mu, \text{ and } \nu = 1, \dots, 4; \mu \neq \nu)$$

$$\text{and } \lambda_\mu - \lambda_\nu = 0,$$

with the relations

$$\sum_i \bar{x}_i = 0$$

$$\sum_i x_i = 0$$

$$\sum_4 \bar{x}_4 = 0$$

$$\text{and } \sum_K x_K \lambda_K = 0.$$

We may take  $x_4 = 1$ , and we then have the point and line of the  $\Gamma_{5,2}^2$  expressed in terms of the three arbitrary constants  $x_1, x_2$ , and  $x_3$ . This is projectively perfectly good.



er, for a direction in which eight constants, and this makes of the eleven constants upon which  $\Gamma_{5,2}^{1,2}$  depends.

The equation of the cone  $\Phi$  of this configuration is

$$\sum_{i=1}^3 \alpha_i (\alpha_i + 1) x_i^2 + 2 \sum_{i=1}^3 \alpha_i \alpha_j \alpha_k x_i x_j x_k = 0$$

$$\text{or } \sum_{i=1}^3 \alpha_i + \alpha_j + \alpha_k = 0 \quad \text{or } \sum_{i=1}^3 l_i^2 - 2 \sum_{i=1}^3 l_i l_j = 0$$

Similarly for the space configuration  $\Gamma_{6,1}^4$ , we may start with the  $xy$  projection  $S_4$  as

$$l_i = 0 \quad (i=1 \dots 6)$$

with the relation

$$\sum_{i=1}^6 \varepsilon_i = 0$$

The complete 6-point with



contain the fifteen lines

$$z_i + \lambda z_j = 0, (i \text{ and } j = 1 \dots 6; i \neq j)$$

the twenty planes

$$x_k + \lambda x_m = 0 \quad (k, m, \text{ and } v = 1 \dots 5;$$

$$\text{and } (1+\lambda)x_k - \lambda x_m - x_v = 0 \quad (k \neq m, m \neq v, v \neq k)$$

and the fifteen  $S_i$ 's

$$x_k = 0$$

$$\text{and } x_k - x_m = 0$$

We cut this figure by the  $S_3$

$$\sum_{K=1}^5 x_K x_K = 0$$

and take as reference tetrahedron in this  $S_3$  the sections of the  $S_i$ :

$$x_1 = 0, x_2 = 0, x_3 = 0, \text{ and } x_4 = 0,$$

and as auxiliary plane the section of the  $S_2$

$$\sum_{K=1}^5 x_K = 0$$

The fifteen points of the  $S_i$ 's



are then

$$x_i \tilde{x}_j - x_j \tilde{x}_i = 0, \quad (i \text{ and } j = 1 \dots 6; i \neq j)$$

the twenty lines are

$$x_K + \lambda x_M = 0 \quad (K, M, \text{ and } V = 1 \dots 5)$$

$$\text{and } (1+\lambda)x_K - \lambda x_M - x_V = 0 \quad (K \neq M, M \neq V, V \neq K)$$

and the fifteen planes are

$$x_K = 0$$

$$x_K - x_M = 0$$

with the relations

$$\sum_i^6 \tilde{x}_i = 0$$

$$\sum_i^6 x_i = 0$$

$$x_5 = 0$$

$$\text{and } \sum_K^5 x_K x_K = 0$$

Making  $x_5 = 1$ , we have four constants  $x_1, x_2, x_3$  and  $x_4$ , which together with the fifteen constants of a space collineation make up the nineteen



constants upon which the general  $\Gamma_{6,2}^{14}$  depends.

The quadric surface  $\Phi$  of this configuration is

$$\sum_{i=1}^4 \alpha_i(\alpha_i+1)x_i^2 + 2 \sum_{i=1}^6 \alpha_i \alpha_j x_i x_j = 0$$

$$\text{or } \sum_{i=1}^4 \frac{\alpha_i + \alpha_j + \alpha_k + 1}{\alpha_i} x_i^2 - 2 \sum_{i=1}^6 \xi_i \xi_j = 0$$

The two dual configurations  $\Gamma_{6,2}^{14}$  and  $C_{6,2}^4$  (which are respectively the section by a plane, and projection on a plane, of the space configuration  $\Gamma_{6,2}^{14}$ ) give rise to certain sets of curves which are worthy of notice.

All the elements of a  $\Gamma_{6,2}^{14}$  containing a given letter



of the six, say  $a$ , form a  $\Gamma_{3,2}^3$ ,  
 with which is connected a curve  
 which we may call  $F_a$ . There  
 are six such curves. The twenty  
 points of a  $\Gamma_{1,2}^{14}$  naturally fall in  
 to ten pairs such as  $abc$  and  $def$ .  
 The point  $abc$  is the pole of the  
 line  $def$  with respect to  $F_a$ ,  
 and since  $def$  lies on  $def_a$ ,  $abc$   
 and  $def$  are conjugate points  
 with respect to  $F_a$ . Similarly,  
 since  $abc$  is the pole of  $bdef$  with  
 respect to  $F_b$ ,  $abc$  and  $def$  are  
 conjugate points with respect  
 to  $F_b$ . In fact, they are con-  
 jugate points with respect  
 to all six  $F$ 's. The same is  
 true of each of the ten pairs  
 of points. Regarding the ten



point pairs is degenerate line  
conics, and the  $F_4$  is point  
conics, we have then ten line  
conics each apolar with each  
of six point conics.

Thus being the case, either  
(1.) the ten conics belong to a  
range and the six to a 4-spread,  
or (2.) the ten belong to a web  
and the six to a Vast, or (3.)  
the ten belong to a 4-spread and  
the six to be pencil. Supposi-  
tion (1.) can not be true, for  
there can be only three degen-  
erate conics in a range. If (2.)  
were true, the twenty point  
would lie on a cubic curve;  
but this is impossible since  
they lie by four on straight



lines. Therefore (3.) is true, and the six conics  $F$  belong to a pencil; i.e., they pass through four points. Thus every  $\Gamma_{5,2}^{v4}$  determines four constant points.

In the same way it may be shown that a  $\Gamma_{5,2}^{v4}$  determines six line conics,  $\Phi_1, \Phi_2, \dots, \Phi_6$ , belonging to a range; and hence every  $\Gamma_{5,2}^{v4}$  determines six constant lines.

Every  $\Gamma_{n,2}^v$  (where  $v \geq 3$  and  $n \geq v+2$ ) contains  $\binom{n-5}{2} \binom{v-3}{1} \Gamma_{5,2}^{v3}$ 's, with each of which is associated a conic  $F$ . We obtain one of these  $\Gamma_{5,2}^{v3}$ 's by picking out all the elements of the  $\Gamma_{n,2}^v$  which contain a certain  $v-3$



letters, say  $a, b, \dots, K$ , and do not contain a certain  $s-v-2$  letters, say  $m, n, \dots, s$ . We may then denote the curve associated with this  $\Gamma_{s,2}^{v-2}$  by the symbol,  $F_{ab\dots K.mn\dots s}$ .

If  $v \geq 4$ , the configuration  $\Gamma_{a,2}^{v-2}$  contains  $\binom{u}{n-6} \binom{u-6}{v-4}$   $\Gamma_{6,2}^4$ 's, with each of which is associated a pencil of six of the curves  $F$ . The  $\Gamma_{6,2}^4$ , and therefore the associated pencil, may be denoted by the symbol  $(b\dots K.mn\dots s)$ , made up of two groups of  $v-4$  and  $s-v-2$  letters. A given curve,  $F_{ab\dots K.mn\dots s}$  belongs to a given pencil,  $(b\dots K.mn\dots s)$ , if the  $s-v-2$  letters in the two symbols are the same, and if the  $v-4$  letters



two in the symbol for the pencil are contained among the  $v-3$  letters in the symbol for the series. We have then, associated with a  $\Gamma_{n,v}^{1,v}$  (where  $v \geq 4$  and  $n \geq v+2$ ),  $\binom{n}{v-5} \binom{n-5}{v-3}$  series lying in  $\binom{n}{v-6} \binom{n-6}{v-4}$  pencils, six lines in each pencil and each series in  $v-2$  pencils.

Similarly, a  $C_{n,v}^m$  (where  $m \geq 4$  and  $n \geq m+2$ ) gives rise to  $\binom{n}{m-5} \binom{n-5}{m-3}$  line series lying in  $\binom{n}{m-6} \binom{n-6}{m-4}$  ranges, six cubics in each range and each series in  $m-3$  ranges.

Restating the last theorem in terms of the  $\Gamma_{n,v}^{1,v}$  instead of  $C_{n,v}^m$ , and substituting  $m = n - v + 1$ , we have —



Associated with every  $\Gamma_{n,2}^v$  (where  $v \geq 3$  and  $n \geq v+3$ ) are  $\binom{n}{n-5} \binom{n-5}{v-3}$  line-conics lying in  $\binom{n}{n-6} \binom{n-6}{v-3}$  ranges, six conics in each range and each conic in  $n-v-2$  ranges.

But these  $\binom{n}{n-5} \binom{n-5}{v-3}$  line-conics are the same as the point-conics,  $F$ , associated with the  $\Gamma_{n,2}^v$ . Hence we have the rather remarkable theorem:—

Associated with every  $\Gamma_{n,2}^v$  (where  $v \geq 4$  and  $n \geq v+3$ ) are  $\binom{n}{n-5} \binom{n-5}{v-3}$  conics which lie by sixes in  $\binom{n}{n-6} \binom{n-6}{v-4}$  pencils, each conic in  $v-3$  pencils, and which also lie by sixes in  $\binom{n}{n-6} \binom{n-6}{v-3}$  ranges, each conic in  $n-v-2$  ranges.



## Vita

I, Walter Buckingham Cannon, was born at Town Hall, Pa., on Jan. 11, 1879. I prepared for college at the Gettysburg College Preparatory School, and I took the first year of my college course at Gettysburg College. I then went to Dickinson College, where I took the degree of B. S. in 1899. I taught mathematics in the Troy Composite Academy, Troy, N. Y., during the academic year of 1899-1900. In October, 1900, I entered the Johns Hopkins University as a graduate student — Mathematics, Phy-



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